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THE BRAUER GROUP OF GRADED AZUMAYA ALGEBRAS

BY

L. N. CHILDS[1], G. GARFINKEL AND M. ORZECH[2]

ABSTRACT. We study G-graded Azumaya R-algebras for R a commutative ring and G a finite abelian group, and a Brauer group formed by such algebras. A short exact sequence is obtained which relates this Brauer group with the usual Brauer group of R and with a group of graded Galois extensions of R. In case G is cyclic a second short exact sequence describes this group of graded Galois extensions in terms of the usual group of Galois extensions of R with group G and a certain group of functions on Spec (R).

Introduction. In [20], C. T. C. Wall introduced the Brauer group of $Z/2Z$-graded central simple algebras over a field K. He called a graded K-algebra $A = A_0 \oplus A_1$ *graded simple* if A had no homogeneous two-sided ideals, and *graded central* if $A_0 \cap \text{center}(A) = K$. Wall's definition resolved an inelegance in the theory of quadratic forms: the Clifford algebra A of a quadratic form $q: V \to K$ is always a *graded* central simple algebra, whereas when V is odd dimensional A is not central simple. Wall showed that a graded central simple K-algebra is described by three invariants: a central simple K-algebra, an element of $Z/2Z$, and a quadratic Galois extension of K (or more precisely, an element of $K/\wp(K)$ if $\text{char}(K) = 2$ and of K^*/K^{*2} if $\text{char}(K) \neq 2$).

H. Bass [2, Chapter IV] and C. Small [16] extended Wall's work to $Z/2Z$-graded algebras over a commutative ring R. They generalized Auslander and Goldman's definition of separable R-algebras to $Z/2Z$-graded R-algebras [1] and defined a Brauer-Wall group of graded central separable R-algebras. This group, $BW(R)$, was described by two exact sequences:

$$0 \to B(R) \to BW(R) \to Q_2(R) \to 0,$$

where $B(R)$ is the usual Brauer group of R and $Q_2(R)$ the group of $Z/2Z$-graded quadratic Galois extensions of R;

$$0 \to \text{Gal}(R, Z/2Z) \to Q_2(R) \to H(R) \to 0,$$

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which relates $Q_2(R)$ to the group $\mathrm{Gal}(R, Z/2Z)$ of quadratic Galois extensions of R (defined by Harrison in [10]), and to $H(R)$, the group of those continuous maps from $\mathrm{Spec}(R)$ to $Z/2Z$ which vanish on primes containing 2.

In §§1 and 2 of this paper we extend the definitions of Bass and Small to R-algebras graded by an abelian group G. However, we do restrict to R being trivially graded. We adopt the device Knus introduced in [14]: graded concepts are defined relative to a fixed bilinear map $\phi: G \times G \to \mathrm{Units}(R)$. We do not assume that ϕ is symmetric or nondegenerate. Surprisingly, to define *graded central separable*, the concept of *graded separable* is unnecessary (cf. Remark (a) following 2.1). Thus our *graded Azumaya R*-algebras A are those separable R-algebras which are G-graded and whose *graded center*, relative to ϕ, is R; the graded center of A is the set of homogeneous x satisfying $xa = \phi(\deg x, \deg a)ax$ and $ax = \phi(\deg a, \deg x)xa$ for all homogeneous a in A. §2 develops the theory of G-graded Azumaya R-algebras. As in the ungraded and $Z/2Z$-graded cases the Morita theorems play an important role. However, we have tried to avoid duplicating the superstructure used in [2] and [1] to prove results about Azumaya R-algebras. It seems likely that one could develop machinery for graded objects which would generalize classical results about ungraded modules and algebras (cf. [16]).

In §3 we define a map π from graded Azumaya R-algebras to graded Galois extensions, and obtain an exact sequence

$$0 \to B(R) \to \mathbf{B}(R, G) \to \mathrm{Im}(\pi) \to 0.$$

In contrast to the case $G = Z/2Z$, $\mathrm{Im}(\pi)$ is not always the full set of graded Galois extensions of R with group G. In fact, the latter set need not be a group under its natural multiplication, but $\mathrm{Im}(\pi)$ is a group.

In §4 we take G to be cyclic of order n. We obtain an exact sequence

$$0 \to \mathrm{Gal}(R, G) \to \mathrm{Im}(\pi) \to H(R) \to 0.$$

$\mathrm{Gal}(R, G)$ denotes the group of Galois extensions of R with group G ([10], [15]). $H(R)$ is a certain set of continuous functions from $\mathrm{Spec}(R)$ to a product of copies of $Z/2Z$, as described in (4.1). This generalizes Bass's exact sequence. As a corollary of our exact sequences, we can deduce Knus's result that for G cyclic of prime order and R an algebraically closed field of characteristic prime to the order of G, $\mathbf{B}(R, G)$ is isomorphic to $Z/2Z$.

1. **Graded algebras and modules.** This section contains basic definitions and results on graded algebras and modules. R will denote a commutative ring, G a finite abelian group. We shall assume the existence of a fixed bilinear form $\phi: G \times G \to \mathrm{Units}(R)$. We shall write G multiplicatively.

An R-algebra A will be called *graded* if $A = \bigoplus_{\sigma \in G} A_\sigma$, where A_σ is an R-submodule of A satisfying $A_\sigma A_\tau \subset A_{\sigma\tau}$ for all τ in G. It follows that $1 \in A_1$.

If A and B are graded R-algebras, an R-algebra homomorphism $f: A \longrightarrow B$ satisfying $f(A_\sigma) \subset B_\sigma$ for σ in G will be called a *homomorphism of graded R-algebras.*

The *graded* tensor product of graded R-algebras A and B is defined as follows: $A \otimes_\phi B = A \otimes_R B$ as R-modules; multiplication is given by

$$(a \otimes b)(c \otimes d) = \phi(b, c)ac \otimes bd,$$

where the notation $\phi(b, c)$ is explained below.

Notational conventions. If x and y are homogeneous elements of the graded R-algebras A and B, with x in A_σ, y in B_τ, we shall write $\phi(x, y)$ instead of $\phi(\sigma, \tau)$. Moreover, the very fact of our writing $\phi(a, b)$ shall imply that a and b are homogeneous elements. Thus, in equations such as the one displayed above, *it is to be understood that the formula extends from homogeneous elements to arbitrary ones by requiring linearity.*

Since ϕ is a fixed pairing, we adopt the device of using boldface notation for constructions or concepts involving ϕ. Thus $A \boldsymbol{\otimes} B$ shall denote $A \otimes_\phi B$, and the very use of $\boldsymbol{\otimes}$ shall imply that we are dealing with graded objects. The grading on $A \boldsymbol{\otimes} B$ is given by $(A \boldsymbol{\otimes} B)_\sigma = \bigoplus_{\alpha\beta=0} (A_\alpha \otimes B_\beta)$.

Let A be a graded R-algebra. By a *graded* left *A-module* we shall mean a left A-module M which decomposes as $M = \bigoplus_{\sigma \in G} M_\sigma$, with $A_\sigma M_\tau \subset M_{\sigma\tau}$ for σ, τ in G.

We define $A^\#$, the graded opposite algebra of A, by $(A^\#)_\sigma = \{a^\# \mid a \in A_\sigma\}$, with $a^\# b^\# = \phi(a, b)(ba)^\#$.

Let $M = \bigoplus M_\sigma$ be a graded (A, B)-bimodule, i.e. $A_\alpha M_\sigma B_\beta \subset M_{\alpha\sigma\beta}$, $(am)b = a(mb)$, and $rm = mr$ for α, β, σ, in G, a in A, m in M, b in B and r in R. We make M a graded $A \boldsymbol{\otimes} B^\#$-module by setting

$$(a \boldsymbol{\otimes} b^\#)m = \phi(b, m)amb$$

for a in A, b in B and m in M. Likewise, a graded left $A \boldsymbol{\otimes} B^\#$-module is a graded (A, B)-bimodule.

Let M and N be graded left A-modules. Define a graded R-module $\mathbf{Hom}_A(M, N)$ by

$$\mathbf{Hom}_A(M, N)_\tau = \{f: M \longrightarrow N \mid f(M_\sigma) \subset N_{\sigma\tau} \text{ for } \sigma \text{ in } G \text{ and}$$
$$f(ax) = \phi(\tau, a)af(x) \text{ for } a \text{ in } A,\ x \text{ in } M\}.$$

$\mathbf{Hom}_A(M, N)$ is the direct sum of the homogeneous pieces thus described. An element of $\mathbf{Hom}_A(M, N)$ will be called an A-**homomorphism.** We shall write A-**Mod** for the category of graded left A-modules and A-**homomorphisms.** If f is homogeneous of degree 1, the condition $f(ax) = \phi(f, a)af(x)$ is just the statement $f(ax) = af(x)$.

We define $\mathbf{A}^e = A \otimes A^{\#}$, ${}^e\mathbf{A} = A^{\#} \otimes A$. We write $A^e = A \otimes_R A^0$ for the usual enveloping algebra of A; A^e is graded via $(A^e)_\sigma = \bigoplus_{\alpha\beta=\sigma}(A_\alpha \otimes_R A^0_\beta)$.

There is an action of $\mathbf{A}^e$ (resp. ${}^e\mathbf{A}$) on A which makes A a graded left (resp. right) module. It is given by $(a \otimes b^{\#})x = \phi(b, x)axb$ (resp. $x(a^{\#} \otimes b) = \phi(x, a)axb$), for a, b and x in A.

There are natural maps, both called π_A, from $\mathbf{A}^e$ and ${}^e\mathbf{A}$ to A, given by deleting #'s and multiplying. These maps are homogeneous of degree 1. We would like to say that these maps are, respectively, an $\mathbf{A}^e$ and ${}^e\mathbf{A}$-module **homomorphism**. For this, and for future reference, we must establish the relationship between left and right actions by graded rings.

Let A be a graded R-algebra, M and N graded right A-modules, i.e. $M_\sigma A_\tau \subset M_{\sigma\tau}$. Define $\mathbf{Hom}_A(M, N)$ to be $\mathrm{Hom}_A(M, N)$ with the induced grading. The fact that the gradings do not affect the definition of homomorphism for right modules, in contrast with the situation for left modules, is consistent with (1.1) below.

Let M, N be graded right modules over the respective graded algebras A and B. It is easily verified that $M \otimes_R N$ is a graded right $A \otimes B$-module via $(m \otimes n)(a \otimes b) = \phi(n, a)ma \otimes nb$. If P is a graded left module over the graded algebra C, then P becomes a graded right $C^{\#}$-module via $pc^{\#} = \phi(p, c)cp$. With these conventions in mind we have

(1.1) Let A and B be graded R-algebras. Let M and N be graded right modules over A and B respectively. Let P and Q be graded left A-modules. Write $\mathbf{E}$ for **End**. Let $C = A \otimes B$. Then

(a) $\mathbf{Hom}_A(P, Q) = \mathbf{Hom}_{A^{\#}}(P, Q)$.

(b) There exists a homomorphism of graded R-algebras,

$$\theta\colon \mathbf{E}_A(M) \otimes \mathbf{E}_B(N) \to \mathbf{E}_C(M \otimes_R N),$$

given by $\theta(\alpha \otimes \beta)(m \otimes n) = \phi(\beta, m)\alpha(m) \otimes \beta(n)$.

For A a graded algebra we shall let **Mod**-A denote the category of graded right A-modules, with morphisms set $\mathbf{Hom}_A(M, N)$. Consider the following situations:

(a) M is in $\mathbf{A}^e$-**Mod**.

(b) N is in **Mod**-${}^e\mathbf{A}$.

Let S be a subset of A, hS the set of homogeneous components of elements of S.

(1.2) **Definition.**

(a)
$$M^S = \{x \in M \mid (s \otimes 1^{\#})x = (1 \otimes s^{\#})x \text{ for } s \in S\} = \bigoplus_\sigma \{x \in M_\sigma \mid sx = \phi(s, x)xs \text{ for } s \in hS\},$$

(b)
$${}^S N = \{x \in N \mid x(1^{\#} \otimes s) = x(s^{\#} \otimes 1) \text{ for } s \in S\} = \bigoplus_\sigma \{x \in N_\sigma \mid xs = \phi(x, s)sx \text{ for } s \in hS\}.$$

(1.3) **Proposition.** *The correspondences* $f \longrightarrow f(1)$ *establish natural isomorphisms*:

(a) $\mathbf{Hom}_B(A, M) = M^A$, $B = \mathbf{A}^e$.

(b) $\mathbf{Hom}_B(A, M) = {}^A M$, $B = {}^e\mathbf{A}$.

When $A = M = S$ we obtain graded R-algebras A^A and ${}^A A$, called respectively the *left* center and *right* center of A.

If M is a graded A-module, we shall write $\mathbf{E}_A(M)$ for $\mathbf{Hom}_A(M, M)$. If $A = R$, then $\mathbf{E}_R(M)$ is simply $\mathrm{End}_R(M)$ as an R-module, but is endowed with a grading. We shall write $E(M)$ for $\mathbf{E}_R(M)$, and $\mathbf{E}_A(M)$ for $\mathbf{End}_A(M)$.

(1.4). **Proposition.** *Let* M *be a graded* R*-module and let* $E = E(M)$. *Then* $E^E = {}^E E = \mathrm{center}(E) = \mathrm{center}(E_1)$.

The graded R-algebras of the type $E(M)$ enjoy a commutativity property vis-a-vis graded tensor products:

(1.5) **Proposition.** *Let* M *be a graded* R*-module and* A *a graded* R*-algebra. Then there exist isomorphisms of graded* R*-algebras.*

$$A \otimes E(M) \cong A \otimes_R E(M) \cong E(M) \otimes A.$$

Proof. Define maps by sending $a \otimes \alpha$ to $a \otimes \phi(\alpha(\ ), a)^{-1}\alpha$, $a \otimes \alpha$ to $\phi(a,\ )^{-1}\alpha \otimes a$. These maps are R-algebra isomorphisms, homogeneous of degree 1. Their respective inverses take $a \otimes \alpha$ to $a \otimes \phi(\alpha(\ ), a)\alpha$ and $\alpha \otimes a$ to $a \otimes \phi(a,\ )\alpha$.

(1.6). **Corollary.** *Let* P *and* Q *be graded projective* R*-modules of finite type. Then the map* $f \otimes g \longrightarrow f \otimes \phi(g(\ ), f)^{-1} g$ *establishes an isomorphism of graded* R*-algebras*

$$E(P) \otimes E(Q) \cong E(P \otimes Q).$$

The following technical results will be useful in §3.

(1.7) Let A and B be graded R-algebras. Then there exists isomorphisms of graded R-algebras

$$(A \otimes B)^{\#} \cong B^{\#} \otimes A^{\#}, \qquad A \cong A^{\#\#}.$$

Proof. Define a map $A \longrightarrow A^{\#\#}$ on homogeneous elements by sending x to $\phi(x, x)x^{\#\#}$. Define a map $B^{\#} \otimes A^{\#} \longrightarrow (A \otimes B)^{\#}$ be sending $b^{\#} \otimes a^{\#}$ to $\phi(b, a)(a \otimes b)^{\#}$. Both maps are easily seen to be R-algebra isomorphisms.

(1.8) **Remark.** Proposition (1.5) will be crucial in defining the Brauer group of graded Azumaya algebras. Except in the case $G = Z/2Z$, the graded tensor product does not yield a commutative operation. Thus (1.5) will be needed to prove that the definition of equivalent Azumaya algebras, following (2.10), yields an equivalence relation compatible with $\otimes$. This point seems to have been overlooked in [14].

2. **Separable algebras and Azumaya algebras.** We begin by analyzing separability for graded algebras:

(2.1) **Theorem.** *Let A be a graded R-algebra. The following conditions are equivalent*:

(a) *A is a projective left $\mathbf{A}^e$-module.*

(b) *There exists an element ϵ' in $\mathbf{A}^e$ satisfying $\pi_A(\epsilon') = 1$ and $(1 \otimes a^{\#})\epsilon' = (a \otimes 1^{\#})\epsilon'$ for all a in A.*

(c) *There exists an element ϵ in $(\mathbf{A}^e)_1$ satisfying $\pi_A(\epsilon) = 1$ and $(1 \otimes a^{\#})\epsilon = (a \otimes 1^{\#})\epsilon$ for all a in A.*

(d) *There exists an element ϵ in $A \otimes_R A^0$ satisfying $\pi_A(\epsilon) = 1$ and $(1 \otimes a^0)\epsilon = (a \otimes 1^0)\epsilon$ for all a in A.*

(e) *There exists an element $'\epsilon$ in ${}^e\mathbf{A}$ satisfying $\pi_A('\epsilon) = 1$ and $'\epsilon(a^{\#} \otimes 1) = {}'\epsilon(1^{\#} \otimes a)$ for all a in A.*

(f) *A is a projective right ${}^e\mathbf{A}$-module.*

Proof. (a) is equivalent to $\pi_A: \mathbf{A}^e \longrightarrow A$ being a split epimorphism, and this is easily seen to be equivalent to (b); if f splits π_A, set $\epsilon' = f(1)$. Since π_A is homogeneous of degree 1, π_A is split by f if and only if π_A is split by f_1, the 1-homogeneous component of f. It is thus clear that (b) is equivalent to (c).

Let $\epsilon = \Sigma x_i \otimes y_i^{\#}$ be in $(A^e)_1$; we may take x_i and y_i homogeneous, with $x_i y_i$ in A_1. The condition $(1 \otimes a^{\#}) = (a \otimes 1^{\#})$ is then equivalent to the equality

$$\sum x_i \otimes (y_i a)^{\#} = \sum a x_i \otimes y_i^{\#} .$$

Noting that there is an R-module isomorphism $\mathbf{A}^e \cong A \otimes_R A^0$, under which $a \otimes b^{\#}$ and $a \otimes b^0$ correspond, we may conclude that (c) is equivalent to (d). Similarly, using the R-module isomorphism $A \otimes_R A^0 = A^0 \otimes_R A$ under which $a \otimes b^0$ and $a^0 \otimes b$ correspond, we conclude that (e) and (f) are equivalent to conditions (a)–(d).

Definition. We say that A is R-**separable** if conditions (a)–(e) above hold.

Remarks. (a) Condition (d) states that A is R-**separable** if and only if it is R-separable in the sense of ungraded algebras.

(b) It is easily computed that ϵ', ϵ and $'\epsilon$, referred to in (b)–(e) above, are idempotents.

(c) Let A be a **separable** R-algebra and choose ϵ' and $'\epsilon$ as in (b) and (e) above. Let M be in $\mathbf{A}^e$-**Mod** and N in **Mod**-${}^e\mathbf{A}$. Then $M^A = \epsilon' M$ and ${}^A N = N'\epsilon$ (cf. (1.2), (1.3)). For if x is in M^A and $\epsilon' = \Sigma x_i \otimes y_i^{\#}$, then $x = \Sigma (x_i y_i \otimes 1^{\#})x = \Sigma (x_i \otimes 1^{\#})(1 \otimes y_i^{\#})x = \epsilon' x$.

(2.2) **Proposition.** (a) *Let* $f: A \longrightarrow B$ *be an onto map of graded* R*-algebras. If* A *is* R-**separable**, *so is* B. *Moreover, in this case* $f(A^A) = B^B$ *and* $f({}^AA) = {}^BB$.

(b) *Let* S *be a commutative* R*-algebra. If* A *is* R-**separable** *then* $B = S \otimes_R A$ *is* S-**separable**. *Moreover, in this case* $B^B = S \otimes_R A^A$ *and* ${}^BB = S \otimes_R {}^AA$.

Proof. Apply the preceding remarks.

(2.3) **Proposition.** (a) *Let* A *be* R-**separable**. *Then* $A^{\#} = B$ *is* R-**separable**. *If in addition* $A^A = R$ *then* ${}^BB = R$; *if* ${}^AA = R$ *then* $B^B = R$.

(b) *Let* A *and* B *be* R-**separable**. *Then* $C = A \otimes B$ *is* R-**separable**. *If* $A^A = R = B^B$ *then* $C^C = R$. *If* ${}^AA = R = {}^BB$ *then* ${}^CC = R$.

Proof. (a) Let $\epsilon = \Sigma\, x_i \otimes y_i^{\#}$ be an element as described in (2.1)(c), with x_i and y_i homogeneous and x_iy_i in A_1. For each a in hA we have

$$(*) \qquad \sum ax_i \otimes y_i^{\#} = \sum x_i \otimes (y_ia)^{\#}.$$

Let $\epsilon^{\#} = \Sigma\, \phi(x_i, x_i)y_i^{\#} \otimes x_i^{\#\#}$. Using the fact that x_iy_i is in A_1, we compute that, for a in hA,

$$(1) \qquad (a^{\#} \otimes 1^{\#\#})\epsilon^{\#} = \sum \phi(a, y_i)\phi(x_i, x_i)(y_ia)^{\#} \otimes x_i^{\#\#},$$

$$(2) \qquad (1^{\#} \otimes a^{\#\#})\epsilon^{\#} = \sum \phi(x_i, ax_i)y_i^{\#} \otimes (ax_i)^{\#\#}.$$

There is an R-module isomorphism $A^{\#} \otimes A^{\#\#} \cong A \otimes A^{\#}$, under which $a^{\#} \otimes b^{\#\#}$ and $\phi(a, b)b \otimes a^{\#}$ correspond. Applying this map to the right-hand sides of (1) and (2), we obtain the two sides of (*). It follows that the left-hand sides of (1) and (2) are equal. It is clear that $\pi_{A^{\#}}(\epsilon^{\#}) = 1$. The rest of the proof of (a) is straightforward.

(b) This argument uses the same idea just employed. We note that there is an R-module isomorphism $A \otimes B \otimes (A \otimes B)^{\#} \cong A \otimes A^{\#} \otimes B \otimes B^{\#}$ under which $a \otimes b \otimes (c \otimes d)^{\#}$ corresponds to $\phi(b, c)a \otimes c^{\#} \otimes b \otimes d^{\#}$. Let $\epsilon_A = \Sigma\, x_i \otimes y_i^{\#}$ and $\epsilon_B = \Sigma\, a_j \otimes b_j^{\#}$ be elements of $\mathbf{A}^e$ and $\mathbf{B}^e$ satisfying (2.1)(c). Let $\epsilon = \Sigma\, \phi(a_j, x_i)\, x_i \otimes a_j \otimes (y_i \otimes b_j)^{\#}$. A straightforward computation such as used above allows us to conclude that ϵ satisfies the conditions required for $A \otimes B$ to be R-separable.

Suppose that $A^A = R = B^B$. By the remark preceding (2.2) we know that $\epsilon_AA = R = \epsilon_BB$. We wish to show that $\epsilon(A \otimes B) = R$. Clearly, $\epsilon(1 \otimes 1) = 1$. Let a and b be homogeneous elements in A and B respectively. Keeping in mind that x_iy_i and a_jb_j are homogeneous of degree 1, we compute that

$$\epsilon(a \otimes b) = \sum_i \phi(y_i, b)\phi(b, y_i)\epsilon_A(a) \otimes \epsilon_B(b).$$

Now $\epsilon_B(b)$ in R implies that $\epsilon_B(b) = 0$ for b not in B_1; if b is in B_1, $\epsilon(a \otimes b) = \epsilon_A(a) \otimes \epsilon_B(b)$ is in R.

We now introduce the objects of our principal interest.

Let A be a graded R-algebra. We shall say that A is *left* **central** if $A^A = R$, *right* **central** if ${}^AA = R$, **central** if $A^A = R = {}^AA$. We shall say that A is a (left, right) **Azumaya** R-algebra if A is R-**separable** and (left, right) **central**.

(2.4) **Example.** Let M be a graded R-module which is faithful, projective and of finite type. Then $\mathrm{End}_R(M)$ is a separable R-algebra with center R [1, Proposition 5.1]. It follows from (1.4) and Remark (a) after (2.1) that $\mathrm{End}_R(M)$ is an Azumaya R-algebra.

Remark. Let A be a left or right Azumaya R-algebra. Then the inclusion map embeds R as a direct summand of A. For let ϵ be as in (2.1)(c). Define $t: A \to A$ by $t(x) = \epsilon x$. For any y in hA we have

$$y(tx) = (y \otimes 1^{\#})\epsilon x = (1 \otimes y^{\#})\epsilon x = \phi(y, tx)(tx)y.$$

Thus tx is in $A^A = R$. The fact that $\pi_A(\epsilon) = 1$ implies that $t(r) = r$ for r in R.

A graded R-algebra A is said to be *graded simple* if A has no homogeneous two-sided ideals except (0) and A.

The next result shows that our Azumaya algebras coincide with those studied by Wall and Knus ([20], [14]) for R a field.

(2.5) **Proposition.** *Let A be a left or right* **Azumaya** *R-algebra. Then A is graded simple if and only if R is a field.*

Proof. Suppose A is graded simple and I is a nonzero ideal of R. Then $IA = A$. Let $t: A \to R$ be a splitting of the inclusion of R in A. Then $R = tA = ItA = I$.

Let A be a **separable** R-algebra, with R a field. Since A is R-separable in the ungraded sense and R-projective, it is an R-module of finite type [19, Proposition 1.1]. Therefore A is a semisimple ring with d.c.c. [2, p. 100, Theorem 3.1].

Let $\mathfrak{A}$ be a two-sided homogeneous ideal of A. There is a central idempotent α in A with $\mathfrak{A} = A\alpha$. Suppose $\alpha = \Sigma\, \alpha_\sigma$ is the decomposition of α into homogeneous components; then α_σ is in $\mathfrak{A}$ by homogeneity and $\alpha_\sigma = \alpha_\sigma \alpha$. Thus $\alpha_\sigma = \Sigma_\tau\, \alpha_\sigma \alpha_\tau$ and $\alpha_\sigma \alpha_\tau = 0$ for $\tau \neq 1$. Similarly, $\alpha\alpha_\sigma = \alpha_\sigma$ implies $\alpha_\tau \alpha_\sigma = 0$ for $\tau \neq 1$. But $\alpha = \alpha^2$ then yields $\alpha = \alpha_1^2$; thus α is homogeneous of degree 1. However, α being central is then equivalent to α being in ${}^AA$ and in A^A. Thus α is in R and $\mathfrak{A} = (0)$ or $\mathfrak{A} = A$.

(2.6) **Lemma.** *Let A be a left or right* **Azumaya** *R-algebra. Then for any maximal homogeneous two-sided ideal $\mathfrak{M}$ of A, we have $\mathfrak{M} = (\mathfrak{M} \cap R)A$, and $\mathfrak{M} \cap R$ is a maximal ideal of R.*

Proof. Let $\mathfrak{m} = \mathfrak{M} \cap R$. By (2.2), $A/\mathfrak{M}$ is a left or right Azumaya $R/\mathfrak{m}$-algebra. Thus $R/\mathfrak{m}$ is a field by (2.5). But $A/\mathfrak{m}A$ is a left or right Azumaya $R/\mathfrak{m}$-algebra, and is therefore graded simple. ($R/\mathfrak{m}$ is embedded in $A/\mathfrak{m}A$ by the remark following (2.4).) Thus $\mathfrak{m}A \subset \mathfrak{M}$ implies that $\mathfrak{m}A = \mathfrak{M}$.

We show next that the Morita equivalences which underlie the theory of Azumaya algebras are valid for **Azumaya** algebras as well.

(2.7) **Lemma.** *Let A be a* **separable** *R-algebra. Let ϵ be a homogeneous element of $\mathbf{A}^e$ (resp. ${}^e\mathbf{A}$) of degree* 1, *satisfying $\pi_A(\epsilon) = 1$ and $(a \otimes 1^{\#})\epsilon = (1 \otimes a^{\#})\epsilon$ (resp. $\epsilon(a^{\#} \otimes 1) = \epsilon(1 \otimes a)$) for all a in A. Then the graded trace ideal, $\mathbf{tr}_B(A)$, where $B = \mathbf{A}^e$ (resp. ${}^e\mathbf{A}$), is $B\epsilon B$. If A is an* **Azumaya** *R-algebra, then $\mathbf{tr}_B(A) = B$.*

Proof. By $\mathbf{tr}_B(A)$ we of course mean the homogeneous ideal of B generated by all $f(x)$, f in $\mathbf{Hom}_B(A, B)$, x in A; as in the ungraded case, $\mathbf{tr}_B(A)$ is a two-sided ideal of B. By (1.3), if f is in $\mathbf{Hom}_B(A, B)$, then $f(x) = \phi(w, x)(x \otimes 1^{\#})w$ (resp. $f(x) = w(1^{\#} \otimes x))$ where $w = f(1)$ is in B^A (resp. ${}^AB$). Then $w = \epsilon u$ (resp. $w = u\epsilon$) by the remark preceding (2.2), so that $f(A) \subset B\epsilon B$. Since ϵ is clearly in $\mathbf{tr}_B(A)$, and the latter is a two-sided ideal, the first result is clear.

Suppose A is an **Azumaya** R-algebra. Then B is an **Azumaya** R-algebra by (2.3). If $B\epsilon B$ is not all of B, (2.6) implies that $B\epsilon B$ is contained in some maximal ideal mB, m a maximal ideal of R. Applying π_A, we see that $A = \pi_A(B\epsilon B) \subset mA$, so that $A = mA$. However, it follows from the remark preceding (2.5) that $mA \cap R = m$, so that $B\epsilon B = B$.

Let A be a graded R-algebra. $E(A)$ denotes $\mathrm{End}_R(A)$. Define homomorphisms of graded R-algebras:

$$\eta_A\colon \mathbf{A}^e \to E(A), \qquad \mu_A\colon {}^e\mathbf{A} \to E(A)^0,$$

by $\eta_A(w)(x) = wx$, $\mu_A(w) = f^0$, where $f(x) = xw$.

(2.8) **Theorem.** *Let A be a graded R-algebra. The following conditions are equivalent.*

(a) *A is an* **Azumaya** *R-algebra.*

(b) *A is a faithful projective R-module of finite type, and η_A, μ_A are isomorphisms.*

(c) $\mathrm{tr}_R(A) = R$, *and η_A, μ_A are isomorphisms.*

(d) *Each of the following pairs of functors establishes an isomorphism of categories*:

(i)
$$G\colon R\text{-}\mathbf{Mod} \to \mathbf{A}^e\text{-}\mathbf{Mod}, \qquad G(X) = A \otimes_R X,$$
$$H\colon \mathbf{A}^e\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}, \qquad H(X) = X^A.$$

(ii)
$$K: R\text{-}\mathbf{Mod} \to \mathbf{Mod}\text{-}{}^eA, \qquad K(X) = \otimes_R A.$$
$$L: \mathbf{Mod}\text{-}{}^e\mathbf{A} \to R\text{-}\mathbf{Mod}, \qquad L(X) = {}^AX.$$

R-**Mod** *denotes the category of R-modules which have a G-grading.*

Proof. (a) $\Rightarrow$ (b). We know from (2.7) that $\mathrm{tr}_B(A) = B$, where $B = \mathbf{A}^e$ or ${}^e\mathbf{A}$. Thus there exist homogeneous elements $f_1, \cdots, f_n$ in $\mathbf{Hom}_B(A, B)$, $x_1, \cdots, x_n$ in A, satisfying $\Sigma\, x_i f_i(1) = 1$. Define $g_i\colon A \to A$ by $g_i(a) = f_i(1)a$. It is easily seen $g_i(A) \subset {}^AA = R$. But then, for x in A, we have $x = \Sigma\, x_i g_i(x)$, proving that A is R-projective of finite type.

That η_A is monic follows from the equalities $w = \Sigma\, wf_i(x_i) = \Sigma\, f_i(wx_i)$ for w in $\mathbf{A}^e$. Now let α be in $E(A)$, and x in A. Since $(1 \otimes a^\#)f_i(1)x = f_i(a)x = (a \otimes 1^\#)f_i(1)x$, we see that $f_i(1)x$ is in $A^A = R$. Let $f_i(1)x = r_i$. Then $\alpha(x) = \Sigma\, \alpha(f_i(x_i)x) = \Sigma\, \alpha(x_i r_i) = \Sigma\, (f_i(1)x)\alpha(x_i)$, which proves that η_A is onto. Likewise, μ_A is onto.

(b) $\Rightarrow$ (c). This is well known.

(c) $\Rightarrow$ (a). The condition $\mathrm{tr}_R(A) = R$ is easily seen to imply two facts: (1) A is projective as an $E(A)$-module and, (2) the natural map $R \to \mathbf{End}_{E(A)}(A)$ is an isomorphism. But (1) implies that A is a projective $\mathbf{A}^e$-module, since μ_A is an isomorphism, and (2) implies that $A^A = R = {}^AA$, by (1.3).

(a)–(c) $\Leftrightarrow$ (d). This can be proved using the computational techniques illustrated by the proof of (a) $\Rightarrow$ (b). At this point enough facts have been established to permit duplication in our context of the arguments in [2, p. 104, Theorem 4.1].

(2.9) **Lemma.** *Let A be any graded R-algebra, M and P graded projective faithful R-modules of finite type. Suppose there is an isomorphism of graded R-algebras, $A \otimes E(M) \cong E(P)$. Then there exists a graded projective faithful R-module of finite type, H, satisfying $A \cong E(H)$ as graded R-algebras.*

Proof. By (1.5), $A \otimes E(M) \cong E(P)$. For ungraded algebras this implies that $A \cong E(H)$, where $H = \mathrm{Hom}_{E(M)}(M, P)$ is a projective R-module of finite type [1, Proposition 5.3]. Since the isomorphism $A \otimes E(M) \cong E(P)$ is homogeneous of degree 1, P is a graded $E(M)$-module and thus H and $E(H)$ are graded. It is easy to see that the isomorphism $A \cong E(H)$ is homogeneous of degree 1.

(2.10) **Proposition.** *Let A and B be* **Azumaya** *R-algebras. Then the following conditions are equivalent*:

(a) *There exist graded projective faithful R-modules of finite type, P and Q, for which $A \otimes E(P) \cong B \otimes E(Q)$ as graded R-algebras.*

(b) *There exists a graded projective faithful R-module of finite type, M, satisfying $A \otimes B^\# \cong E(M)$ as graded R-algebras.*

Proof. This result follows easily by use of (1.5), (2.9) and the well-known formula $E(P \otimes Q) \cong E(P) \otimes E(Q)$.

Two algebras related as in (2.10) will be called **equivalent**. It is easily verified that this yields an equivalence relation compatible with $\otimes$; (1.5) plays a crucial role. The equivalence classes of Azumaya R-algebras form a group. Multiplication is given by $(A)(B) = (A \otimes B)$; If P is a graded projective faithful R-module, $E(P)$ represents the identity element; the inverse of (A) is $(A^{\#})$. This group is denoted by $\mathbf{B}(R, G)$, and is called the Brauer group of graded Azumaya R-algebras.

A graded R-algebra A will be called *fully graded* if, for each σ in G, $A_\sigma A_{\sigma^{-1}} = A_1$. This is equivalent to having $A_\sigma A_\tau = A_{\sigma\tau}$ for σ, τ in G. If B is fully graded, so is $A \otimes B$. Since $E(RG)$ is easily seen to be fully graded (RG is the group ring) we see that *every element in* $\mathbf{B}(R, G)$ *has a fully graded representative* Azumaya R-algebra.

(2.11) Let A be a fully graded R-algebra. Then each A_σ is a projective (left and right) A_1-module of finite type. If in addition A is R-separable, then A_1 is R-separable.

Proof. Fix σ in G, and choose $x_1, \cdots, x_n$ in $A_{\sigma^{-1}}$, $y_1, \cdots, y_n$ in A_σ satisfying $\Sigma\, x_i y_i = 1$. Define $f_i: A_\sigma \to A_1$ by $f_i(x) = xx_i$; f_i is a homomorphism of left A_1-modules. For x in A_σ, $x = \Sigma\, f_i(x)y_i$. Thus $\{y_i, f_i\}$ is a projective coordinate system for A_σ as a left A_1-module. Similarly, $\{x_i, y_i(\ )\}$ is a projective coordinate system for $A_{\sigma^{-1}}$ as a right A_1-module.

Since A is A_1-projective, $A \otimes_R A^0$ is $A_1 \otimes_R A_1^0$-projective. But R-separability of A implies that A is an $A \otimes_R A^0$-direct summand, and thus an $A_1 \otimes_R A_1^0$-direct summand, of $A \otimes_R A^0$. Since A_1 is an $A_1 \otimes_R A_1^0$-direct summand of A, it follows that A_1 is $A_1 \otimes_R A_1^0$-projective, and therefore R-separable.

Remark. Let A be fully graded and R-separable. The separability of A_1 implies that there exists an element e in $A_1 \otimes_R A_1^0$ satisfying $(1 \otimes a^0)e = (a \otimes 1^0)e$ for a in A_1. Let M be any A^e-module. It is clear that eM is a subset of M^{A_1}. (We are identifying $a \otimes b^0$ with $a \otimes b^{\#}$ when a, b are in A_1.) If e also satisfies $\pi_{A_1}(e) = 1$, then $eM = M^{A_1}$ (cf. Remark (c) following (2.1)).

Now let A and B be graded R-algebras with A_1 and B_1 separable. Let e_A in A_1^e, e_B in B_1^e be elements as above. Then $e_A \otimes e_B$ is identified in an obvious way with an element $e_{A\otimes B}$ of $(A_1 \otimes_R B_1)^e$ which guarantees separability of $A_1 \otimes_R B_1$. From the above discussion, $(A \otimes B)^{A_1 \otimes B_1} = e_{A\otimes B}(A \otimes B) = e_A A \otimes e_B B$. Therefore

$$(2.12) \qquad (A \otimes B)^{A_1 \otimes_R B_1} = A^{A_1} \otimes B^{B_1}.$$

3. **Azumaya algebras and Galois extensions.** Given an Azumaya R-algebra A, we wish to associate to A a Galois extension of R with group G. We shall describe two equivalent ways of doing this.

For S any ring, let GS denote the collection of set maps from G to S, a ring under pointwise operations. If A is a graded R-algebra, so is GA; $(GA)_\sigma$ consists of those maps ν with $\nu(G) \subset A_\sigma$. Define a left A^e-module structure on GA by setting

$$[(a \otimes b^{\#})d](\sigma) = \phi(b,\ d)ad(\sigma b)b,$$

for $a,\ b,\ d$ homogenous elements (so that σb makes sense). Define $\Gamma(A) = (GA)^A$.

There is a natural action of G on GA: set $(\sigma d)(\tau) = d(\sigma^{-1}\tau)$. Then G acts as a group of R-algebra automorphisms of GA, and by restriction to $\Gamma(A)$, G acts as a group of R-algebra automorphisms of $\Gamma(A)$, each homogeneous of degree 1.

The R-algebra A is embedded in GA as the set of constant functions, and in fact $A = (GA)^G$, the set of G-invariant elements. It is then easily checked that $\Gamma(A)^G = A^A$. The next result holds.

(3.1) If A is central then $\Gamma(A)^G = R$.

Let A be an Azumaya R-algebra. For $\Gamma = \Gamma(A)$, define $\theta: \Gamma \otimes_R \Gamma \to G\Gamma$ by $\theta(x \otimes y)(\sigma) = x\sigma(y)$. Then θ is an isomorphism: By (2.8)(d), we have an isomorphism $A \otimes \Gamma \cong GA$ (unadorned tensor products are over R). We need to know this isomorphism explicitly: an examination of our proof that (a)–(c) $\Rightarrow$ (d) in (2.8), and of [2, p. 104, Theorem 4.1], shows that the above map is determined by sending $a \otimes d$ to ad. It is easily verified that η: $GA \otimes_A GA \to G(GA)$, defined analogously to θ, is an isomorphism. We have a commutative diagram as below, from which it follows that $1 \otimes \theta$, and therefore θ, is an isomorphism.

$$\begin{array}{ccc}
A \otimes \Gamma \otimes \Gamma & \xrightarrow{1\otimes\theta} & A \otimes G\Gamma \\
\cong\big\downarrow & & \big\downarrow\cong \\
GA \otimes \Gamma & & G(A \otimes \Gamma) \\
\cong\big\downarrow & & \\
GA \otimes_A A \otimes \Gamma & & \big\downarrow\cong \\
\cong\big\downarrow & & \\
GA \otimes_A GA & \underset{\eta}{\xrightarrow{\cong}} & G(GA).
\end{array}$$

Using [4, Theorem 1.3], we obtain (3.2) below. It should be noted that the proof in [4, Theorem 1.3], (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) does not require A to be commutative.

(3.2) $\Gamma(A) = (GA)^A$ is a Galois extension of R with group G. Moreover, for σ in G and $\Gamma(A)_\tau$ the τ-homogeneous component of $\Gamma(A)$, $\sigma(\Gamma(A)_\tau) \subset \Gamma(A)_\tau$.

We shall find it useful to have a different characterization of $\Gamma(A)$. Let A be a fully graded Azumaya R-algebra. Then $A_{\sigma^{-1}}A_\sigma = A_1$ implies that, for σ in G, there exist elements $b_{\sigma,i}$ in $A_{\sigma^{-1}}$, $c_{\sigma,i}$ in A_σ, satisfying $\Sigma_i b_{\sigma,i}c_{\sigma,i} = 1$.

Let $\pi(A) = A^{A_1}$. Define an action of G on $\pi(A)$ by

$$\sigma(a) = \phi(\sigma, a)\sum_i b_{\sigma,i}ac_{\sigma,i}.$$

This action is independent of the choice of $b_{\sigma,i}$ and $c_{\sigma,i}$. For let $\Sigma_i d_{\sigma,i}e_{\sigma,i} = 1$. Then $\Sigma_i b_{\sigma,i}ac_{\sigma,i} = \Sigma_{i,j} b_{\sigma,i}ac_{\sigma,i}d_{\sigma,j}e_{\sigma,j} = \Sigma_{i,j} b_{\sigma,i}c_{\sigma,i}d_{\sigma,j}ae_{\sigma,j}$, since a is in $\pi(A)$. A similar computation shows that $\sigma(a)\sigma(b) = \sigma(ab)$, and that σ is an isomorphism with inverse σ^{-1}.

For a in $\pi(A)$ and x in A_σ, define $a^x = \sigma(a)$. We then have that $xa^x = \phi(x, a)x\,\Sigma\, b_{\sigma,i}ac_{\sigma,i} = \phi(x, a)ax\,\Sigma\, b_{\sigma,i}c_{\sigma,i} = \phi(x, a)ax$. It follows that

$$\pi(A) = \{a \text{ in } A \mid xa^x = \phi(x, a)ax\}. \tag{3.3}$$

It follows easily that $\pi(A)^G = A^A = R$. The $b_{\sigma,i}$ and $c_{\sigma,i}$ allow us to compute $\Gamma(A)$ more explicitly. It is straightforward to verify that d is in $(GA)^A$ if and only if $a_\rho d(\sigma\rho^{-1}) = \phi(\rho, d(\sigma))d(\sigma)a_\rho$ for σ, ρ in G, a_ρ in A_ρ and d homogeneous. In particular, choose $a_\rho = c_{\rho,i}$, multiply on the left by $b_{\sigma,i}$ and obtain

$$d \text{ in } (GA)^A \Rightarrow d(\sigma\rho^{-1}) = \rho d(\sigma).$$

It is an easy consequence of the above characterization of $(GA)^A$ that $d(1)$ is in $\pi(A)$, and therefore:

$$\Gamma(A) = \{d \mid d(\sigma) = \sigma^{-1}(c),\ c \in \pi(A)\}.$$

Define maps $\pi(A) \to \Gamma(A)$, $\Gamma(A) \to \pi(A)$ by $c \to \Sigma_\sigma \sigma^{-1}(c)e_\sigma$, where $e_\sigma(\tau) = \delta_{\sigma,\tau}$ and $d \to d(1)$. These are inverse isomorphisms of graded R-algebras, and are RG-module maps as well. We summarize these facts:

(3.4) **Proposition.** *Let A be a fully graded Azumaya R-algebra. Then $(GA)^A$ and A^{A_1} are isomorphic Galois extensions of R with group G. The elements of G act as isomorphisms of graded R-algebras.*

(3.5) **Remark.** The set of isomorphism classes of Galois extensions of R with group G, $\mathrm{Gal}(R, G)$, is endowed with a group structure [15, § 1]. The trivial element of this group is GR, and the multiplication is defined by $(S)\cdot(T) = (S \otimes_R T)^{\Delta G}$, where ΔG consists of the elements (σ, σ^{-1}) in $G \times G$ [15, p. 486].

Let $\mathrm{Gal}(R, G)$ denote the set of isomorphism classes of Galois extensions of R with group G. We would like to show that the correspondence $A \to \Gamma(A)$ determines a map from $\mathrm{B}(R, G)$ to $\mathrm{Gal}(R, G)$. We shall first consider a special case of this problem.

Let A and B be fully graded Azumaya R-algebras. It is obvious that $\pi(A \otimes B) \subset (A \otimes B)^{A 1 \otimes B 1}$, and by (2.12), $\pi(A \otimes B) \subset \pi(A) \otimes \pi(B)$. The G-actions on $\pi(A)$ and $\pi(B)$ induce a $G \times G$-action on $\pi(A) \otimes \pi(B)$, since G acts as homogeneous automorphisms in each case, $(\sigma, \tau)(a \otimes b) = \sigma a \otimes \tau b$. We shall write $[\pi(A) \otimes \pi(B)]^{\Delta G}$ for the set of elements invariant under all automorphisms (σ, σ^{-1}), σ in G. This R-algebra obtains a well-defined G-action via $\sigma x = (\sigma, 1)x$.

(3.6) **Proposition.** *There is an equality of graded Galois extensions*

$$\pi(A \otimes B) = [\pi(A) \otimes \pi(B)]^{\Delta G}$$

provided that $\phi(\sigma, \alpha)\phi(\alpha, \sigma) = 1$ *for all* σ *in* G *and for those* α *in* G *satisfying* $\pi(B)_\alpha \neq (0)$.

Proof. Let w be an element of $\pi(A \otimes B)$. By the remarks above w is in $\pi(A) \otimes \pi(B)$, and for ease of computation write $w = x \otimes y$, x in $\pi(A)$, y in $\pi(B)$, each homogeneous. Let b_i in $A_{\sigma^{-1}}$, c_i in A_σ, β_j in B_σ, γ_j in $B_{\sigma^{-1}}$ be elements satisfying

$$\sigma(u) = \phi(\sigma, x) \sum_i b_i u c_i, \qquad \sigma^{-1}(v) = \phi(\sigma^{-1}, y) \sum_j \beta_j v \gamma_j,$$

for u in $\pi(A)$, v in $\pi(B)$. A straightforward computation yields

$$(\sigma, \sigma^{-1})(x \otimes y) = \alpha \sum_{i,j} (b_i \otimes \beta_j)(x \otimes y)(c_i \otimes \gamma_j),$$

where $\alpha = \phi(\sigma^{-1}, y)\phi(y, \sigma^{-1})\phi(\sigma, \sigma^{-1})$. But $b_i \otimes \beta_j$ is in $(A \otimes B)_1$, and therefore commutes with $x \otimes y$; recall moreover that $\Sigma_i b_i c_i = 1$, $\Sigma_j \beta_j \gamma_j = 1$. We therefore conclude that

$$(*) \qquad (\sigma, \sigma^{-1})(x \otimes y) = \phi(\sigma^{-1}, y)\phi(y, \sigma^{-1})(x \otimes y),$$

and the desired equality of sets is clear. It is a straightforward computation that the G-action naturally obtained on $\pi(A \otimes B)$ agrees with the G-action induced on $[\pi(A) \otimes \pi(B)]^{\Delta G}$; indeed one may choose $(b_i \otimes 1)$ in $(A \otimes B)_\sigma$, $(c_i \otimes 1)$ in $(A \otimes B)_{\sigma^{-1}}$ as the elements which determine the G-action on $\pi(A \otimes B)$. This concludes the proof of the Proposition.

Let $P = \bigoplus P_\sigma$ be a graded R-module, with each P_σ a finitely generated faithful projective R-module. Let $E = \mathrm{End}_R(P)$, and assume that E is fully graded. By (2.4), E is an Azumaya R-algebra. We claim

(3.7) $E^{E_1} \cong GR$, with trivial grading.

Let f be an element of E^{E_1}, homogeneous of degree σ. Let h_τ denote the element of E_1 which is the identity on P_τ, zero elsewhere. Then $fh_\tau = h_\tau f$ implies that f is homogeneous of degree 1. Write $f = \Sigma f^\sigma$, with f^σ in $\mathrm{End}_R(P_\sigma)$.

Clearly f is in the center of $\mathrm{End}_R(P_\sigma)$, which is R [1, Proposition 5.1].

Let P remain as above, and let A be a fully graded **Azumaya** R-algebra. By (3.6) and (3.7), $\pi(A \otimes E) = (\pi(A) \otimes GR)^{\Delta G} = (\pi(A) \otimes GR)^{\Delta G}$. The usual isomorphism from $\pi(A)$ to $(\pi(A) \otimes GR)^{\Delta G}$ given by $a \mapsto \Sigma\, \sigma(a) \otimes v_{\sigma^{-1}}$ yields an isomorphism from $\pi(A \otimes E)$ to $\pi(A)$ as graded Galois extensions.

Now suppose A and B are fully graded Azumaya R-algebras in the same equivalence class in $\mathbf{B}(R, G)$. By (2.10), $A \otimes E(M) \cong B \otimes E(N)$, with M, N faithful projective R-modules of finite type. Tensor both sides with $E(RG)$. Using (1.6), we can conclude that $A \otimes E(P) \cong B \otimes E(Q)$ where P and Q have the nice properties that our module P discussed above was assumed to possess. It then follows that $\pi(A)$ and $\pi(B)$ are isomorphic as graded Galois extensions.

Let $\mathbf{Gal}(R, G)$ denote the set of isomorphism classes of graded Galois extensions, i.e. of graded R-algebras which are Galois extensions of R, with group G and on which G acts as a group of R-algebra automorphisms, each homogeneous of degree 1. In taking isomorphism classes, only maps of degree one are considered. Then

(3.8) π determines a map from $\mathbf{B}(R, G)$ to $\mathbf{Gal}(R, G)$. $\pi(A)$ is the trivial element of $\mathbf{Gal}(R, G)$, viz. (GR), if and only if the equivalence class of A contains a trivially graded central separable R-algebra.

Proof. Only the second statement remains to be proved. Let A be an ungraded central separable R-algebra; $D = A \otimes E(RG)$ is a fully graded Azumaya R-algebra in the equivalence class of A. Then $\pi(A) = \pi(D) = (R \otimes GR)$ by (2.12) and (3.7).

Conversely, suppose $\pi(A) = (GR)$. Then there exist pairwise orthogonal idempotents e_σ in $\pi(A)$, σ in G, with $\Sigma_\sigma\, e_\sigma = 1$. Note that since $\pi(A)$ is trivially graded, $\pi(A)$ is a subset of A_1. Our G-action was defined so that $\sigma(e_\tau) = e_{\sigma\tau}$. From (3.3) we conclude that $xe_\sigma = e_1 x$ for x in A_σ. It follows that $B = e_1 A e_1 = \bigoplus_\sigma A_\sigma e_\sigma e_1 = A_1 e_1$, and is an ungraded R-algebra. In fact, we shall show that B is an **Azumaya** R-algebra, with $(B) = (A)^{-1}$ in $\mathbf{B}(R, G)$.

Consider the graded left A-module Ae_1. It is a projective A-module of finite type. Define a map $\mathbf{Hom}_A(Ae_1, A) \to e_1 A$ by sending f to $f(e_1)$; $\mathbf{Hom}$ is as defined proceding (1.1). Since e_1 is in $\pi(A)$, and thus in A_1, this map is well defined. It is easily verified that this map is an isomorphism, whose inverse sends x to $\phi(\ ,x)(\ )x$. It follows that the graded trace ideal, $\mathbf{tr}_A(Ae_1)$ is Ae_1A (cf. Proof of (2.7)). But $Ae_1A = A$, as is easily verified from the fact, mentioned above, that $A_\sigma e_\sigma = e_1 A_\sigma$. Thus $\mathbf{tr}_A(Ae_1) = A$. A computation similar to the one carried out above shows that $\mathbf{End}_A(Ae_1) = (e_1 A e_1)^0$.

Write $P = Ae_1$, $Q = A(1 - e_1)$. These are graded projective left A-modules, and as explained preceding (1.2), may be viewed as graded right $A^\#$-modules.

Also, viewing A as a graded right A-module yields that $A = \mathbf{End}_A(A)$. Write C for $A^{\#} \otimes A = {}^{e}\mathbf{A}$. From (1.2) we have a map of graded R-algebras:

$$\theta\colon \mathbf{E}_{A^{\#}}(P) \otimes \mathbf{E}_A(A) \to \mathbf{E}_C(P \otimes_R A).$$

This map is in fact an isomorphism: for $A = P \oplus Q$, and θ would be an isomorphism were A substituted for P. Using (1.2), we have that there exists an isomorphism of graded R-algebras

$$\mathbf{E}_A(P) \otimes A \cong \mathbf{E}_C(P \otimes_R A).$$

Because of the category isomorphism noted in (2.8)(d)(ii), there is an isomorphism of graded R-algebras,

$$\mathbf{E}_C(P \otimes_R A) \cong E(M) \quad \text{where } M = {}^{A}(P \otimes_R A).$$

Because P is A-projective and $\mathbf{tr}_A(P) = A$, it follows that $P \otimes_R A$ is C-projective and $\mathbf{tr}_C(P \otimes_R A) = C$. But then M is a graded projective R-module of finite type, and is faithful. Thus $E(M)$ represents the trivial element of $\mathbf{B}(R, G)$. It remains only to show that $\mathbf{E}_A(P)$ is an Azumaya R-algebra. As noted earlier, $\mathbf{E}_A(P)$ is concentrated in degree 1; thus we have an isomorphism of ungraded R-algebras

$$\mathbf{E}_A(P) \otimes A \cong E(M).$$

It follows from [2, Theorem 4.1, Condition (6), p. 105] that $\mathbf{E}_A(P)$ is an **Azumaya** R-algebra. This completes the proof of (3.8).

We now turn our attention to $\mathrm{Gal}(R, G)$ and to the image of π. Let S be a graded Galois extension of R with group G. Define a G-action on $S^{\#}$ by $\sigma(S^{\#}) = (\sigma^{-1}s)^{\#}$. It is easily verified that $S^{\#}$ is a Galois extension, using criterion (b) of [4, Theorem 1.3]: If $x_1, \cdots, x_n, y_1, \cdots, y_n$ are homogeneous elements of S satisfying $\sum_i x_i\sigma(y_i) = \delta_{1,\sigma}$, then $\phi(y_i, x_i)^{-1}y_i^{\#}, x_i^{\#}$, $i = 1, \cdots, n$, are corresponding elements of $S^{\#}$; the fact that $(S^{\#})^G = R$ follows from $S^G = R$.

If S and T are graded Galois extensions with group G, then $S \otimes T$ is a graded Galois extension with group $G \times G$: The existence of elements u_j, v_j satisfying $\sum_j u_j(\sigma, \tau)v_j = \delta_{1,\sigma}\delta_{1,\tau}$ is a consequence of the existence of similar elements in S and T; that $(S \otimes T)^{G\times G} = R$ is derivable from the fact that $\mathrm{trace}_{G\times G}(S \otimes T) = R$ [4, Lemma 1.6].

Let S be a graded Galois extension with group G, and let H be a subgroup of G. Then S^H is a graded Galois extension of R with group G/H. This is easily derived using, e.g. [8, Lemma 1], (cf. [4, Theorem 2.2], [17, Proposition 1]).

Let $\Delta G = \{(\sigma, \sigma^{-1}) \mid \sigma \text{ in } G\}$. Putting together the preceding three paragraphs, we conclude that

(3.9) $(S \otimes S^{\#})^{\Delta G}$ is a graded Galois extension of R with Galois group G.

We wish to show that if $\pi(A) = \pi(B)$ for A and B fully graded **Azumaya** R-algebras, then $\pi(A \otimes B^{\#}) = GR$, with trivial grading. First, we investigate the structure of $\pi(B^{\#})$ as a Galois extension. It is clear that as a graded R-algebra, $\pi(B^{\#}) = \pi(B)^{\#}$. To determine the G-action on $\pi(B^{\#})$, one first chooses elements $b_{\sigma,i}$ in $B_{\sigma^{-1}}$, $c_{\sigma,i}$ in B_σ for which $\sigma(b) = \phi(\sigma, b)\sum_i b_{\sigma,i} b c_{\sigma,i}$ for b in $\pi(B)$. It is then a straightforward matter to compute that

(3.10) The action of G on $\pi(B^{\#})$ is given by $\sigma(x^{\#}) = \phi(\sigma,x)\phi(x,\sigma)(\sigma^{-1}x)^{\#}$ for $x^{\#}$ a homogeneous element of $\pi(B^{\#}) = \pi(B)^{\#}$.

As noted prior to (3.6), $\pi(A \otimes B^{\#}) \subset \pi(A) \otimes \pi(B^{\#})$. The formula $(*)$, derived in the proof of (3.6), implies that $\Sigma\, a_i \otimes b_i^{\#}$ is in $\pi(A \otimes B^{\#})$ if and only if

$$(\sigma, \sigma^{-1}) \sum_i a_i \otimes b_i^{\#} = \sum_i \phi(\sigma^{-1}, b_i)\phi(b_i, \sigma^{-1}) a_i \otimes b_i^{\#}.$$

Considering the $G \times G$-homogeneous components of $\pi(A) \otimes \pi(B^{\#})$, and using (3.10) to compute $\sigma^{-1}(b^{\#})$, we obtain

(3.11) An element $\Sigma_i a_i \otimes b_i^{\#}$, a_i in $\pi(A)$, b_i in $\pi(B)$, is in $\pi(A \otimes B^{\#})$ if and only if $\Sigma_i a_i \otimes b_i^{\#} = \Sigma_i \sigma(a_i) \otimes (\sigma b_i)^{\#}$ for all σ in G.

Now suppose $\pi(A) = \pi(B) = S$. It is clear from (3.11), and from the discussion preceding (3.9), that we have an equality of graded Galois extensions, $\pi(A \otimes B^{\#}) = (S \otimes S^{\#})^{\Delta G}$.

(3.12) Let S be a graded Galois extension of R with group G. There is an isomorphism of graded Galois extensions,

$$(S \otimes S^{\#})^{\Delta G} = GR,$$

where GR is trivially graded. If A and B are fully graded Azumaya R-algebras with $\pi(A) = \pi(B)$, then $A \otimes B^{\#}$ is equivalent to a trivially graded central separable R-algebra.

Proof. The last statement follows from the first statement, the paragraph preceding (3.12), and (3.8). For the first statement, let $\Sigma_i s_i \otimes t_i^{\#}$ be in $(S \otimes S^{\#})^{\Delta G}$; this is equivalent to having $\Sigma_i s_i \otimes t_i^{\#} = \Sigma_i \sigma s_i \otimes (\sigma t_i)^{\#}$ for all σ in G. Note also that because the G-action on $(S \otimes S^{\#})^{\Delta G}$ is via either factor of $G \times G$, it follows that the map j defined below is indeed a map into GR:

$$j: (S \otimes S^{\#})^{\Delta G} \to GR$$

by $j(\Sigma_i s_i \otimes t_i^{\#})(\tau) = \Sigma_i s_i \tau(t_i)$. For w an element of $(S \otimes S^{\#})^{\Delta G}$, let w_1 be the 1-homogeneous component of w. It is clear that $j(w) = j(w_1)$, since $\Sigma_i s_i \tau(t_i)$ is in $R \subset S_1$. It is then a straightforward matter to compute that j is an R-algebra homomorphism, and a G-module map as well; recall that G acts on GR by

$\sigma v(\tau) = v(\sigma^{-1}\tau)$. As a consequence of [4, Theorem 3.4] or [8, Proposition 1], j is an isomorphism. As we remarked above, the restriction of j to the 1-homogeneous component of $(S \otimes S^{\#})^{\Delta G}$ is already onto GR since j is onto GR. It follows that $(S \otimes S^{\#})^{\Delta G}$ is trivially graded. This concludes the proof.

We now state the main result of this section:

(3.13) **Theorem.** *There exists an exact sequence of groups*

$$1 \to B(R) \xrightarrow{\iota} \mathbf{B}(R, G) \xrightarrow{\pi} \operatorname{Im}(\pi) \to 1,$$

where $\iota(A) = A$ *and* $\pi(B) = B^{B_1}$ *for* B *a fully graded* Azumaya *R-algebra.*

Proof. It is easily seen that ι is a homomorphism into the center of $\mathbf{B}(R, G)$ and is monic by (2.9). From (3.12) it follows that if $\pi(A) = \pi(B)$ then there exist trivially graded Azumaya R-algebras A_0 and B_0 with $A \otimes A_0$ and $B \otimes B_0$ in the same equivalence class in $\mathbf{B}(R, G)$. The converse follows from (3.6) and (3.8). Thus π induces a bijection between $\mathbf{B}(R, G)/\iota B(R)$ and the image of π in $\mathbf{Gal}(R, G)$. Giving $\operatorname{Im}(\pi)$ the induced multiplication clearly makes π into a homomorphism whose kernel is the image of ι.

(3.14) **Remarks.** Although the multiplication on $\operatorname{Im}(\pi)$ is defined formally, Proposition (3.6) describes the multiplication explicitly in certain situations, and provides enough information to enable us to obtain, in §4, a precise description of $\operatorname{Im}(\pi)$ when G is cyclic.

We now indicate, without giving detailed proofs, that the image of π and the multiplication on it can be made explicit when the following conditions hold: (i) The only idempotents in R are 0 and 1, (ii) there exists a primitive mth root of 1 in R, where m is the exponent of G and (iii) the order of G, call it n, is a unit in R. Let G^* denote $\operatorname{Hom}(G, U(R))$. The above assumptions imply that $G \cong G^*$ and that the following relations hold for σ in G and χ in G^* ([18, p. 178], [11, Corollary 2.5]):

$$\sum_{\chi \in G^*} \chi(\sigma) = \delta_{1,\sigma}, \qquad \sum_{\sigma \in G} \chi(\sigma) = \delta_{1,\chi}. \tag{3.15}$$

For A a fully graded Azumaya R-algebra there exist $x_{\sigma,i}$ in $A_{\sigma^{-1}}$, $y_{\sigma,i}$ in A_σ satisfying $\Sigma_i\, x_{\sigma,i} y_{\sigma,i} = 1/n$. Define an action of G^* on A by $\chi(a) = \chi(\sigma)a$ for a in A_σ. Using the ideas of [4, Theorem 1.3], it can be shown that the existence of $x_{\sigma,i}$ and $y_{\sigma,i}$ imply that the map $j\colon D(A, G^*) \to \operatorname{End}_{A_1}(A)$ is an isomorphism; $D(A, G^*)$ is the trivial crossed product, A is viewed as a right A_1-module, and $j(au_\chi)(b) = a\chi(b)$. Thus A is a Galois extension of A_1 with group G^*, according to Kanzaki's definition [12]. Define

$$A_{\sigma,\chi} = \{a \in A_\sigma \mid ba = \phi(\tau, \sigma)\chi(\tau)ab \text{ for } b \text{ in } A_\tau\}.$$

It is clear that $A_{\sigma,\chi} \subset \pi(A)$. In fact one can obtain that $\pi(A) = \bigoplus_{\sigma,\chi} A_{\sigma,\chi}$. This is accomplished by noting that $h: \pi(A) \to \mathrm{End}_{A_1}(A)^A$ via $h(x)(y) = \phi(y, x)^{-1}yx$ is an R-module isomorphism, and thus $\pi(A) = \mathrm{End}_{A_1}(A)^A \cong D(A, G^*)^A$.

Each A_χ is a projective R-module of rank 1: for, the argument that $\pi(A) \cong D(A, G^*)^A$ shows implicitly that $A_\chi \cong (Au_\chi)^A$ and one may then apply (2.8)(d). It follows, as in [5, Theorem 1], that $A_\chi A_\theta = A_{\chi\theta}$.

Define an action of G on $\pi(A)$ by $\sigma(a) = \chi(\sigma)a$ for a in A_χ. This agrees with our usual action on $\pi(A)$. The fact that n is a unit can be used to obtain a direct proof that $\pi(A)$ is a Galois extention of R with group G.

Since A_χ is projective of rank 1, there exists a unique σ for which $A_{\sigma,\chi} \neq 0$, and $A_{\sigma,\chi} = A_\sigma$. A thus determines a homomorphism $f_A: G^* \to G$. If A and B are fully graded Azumaya R-algebras, the product $\pi(A) \cdot \pi(B)$ is determined by the following observation: for θ in G^*, $(A \otimes B)_\theta = A_{\theta'} \otimes B_\theta$ where

$$\theta'(\alpha) = \phi(f_B(\theta), \alpha)\phi(\alpha, f_B(\theta))\theta(\alpha).$$

The above decomposition of our Galois extensions relates to the ones of [13] and [5] (cf. (4.2)).

4. **The image of** π. In §3 we showed that there is an exact sequence of groups

$$0 \to B(R) \to \mathbf{B}(R, G) \to \mathrm{Im}(\pi) \to 1,$$

where, for A a fully graded Azumaya R-algebra, $\pi(A) = A^{A_1}$. In this section we describe $\mathrm{Im}(\pi)$ for G a finite cyclic group. Note that for G a cyclic group the bilinear map ϕ is necessarily symmetric, being determined by $\phi(\sigma, \sigma)$ for σ a generator of G. In addition, $\pi(A)$ is commutative [7, Theorem 11].

Let ϕ be a bilinear map on a group G. We shall say that ϕ is *degenerate* on G if there exists $\sigma \neq 1$ in G satisfying $\phi(\sigma, \tau) = 1$ for all τ in G; otherwise ϕ will be said to be *nondegenerate* on G.

Let p be in $\mathrm{Spec}(R)$ and f in $\mathrm{Cont}(\mathrm{Spec}(R), (Z/2Z)^r)$, the set of continuous functions from $\mathrm{Spec}(R)$ to the r-fold product of $Z/2Z$. Write $f(p)_i$ for the projection of $f(p)$ on the ith copy of $Z/2Z$.

(4.1) **Theorem.** *Let $G = \prod_{i=1}^r G_i$ be a cyclic group of order n, with G_i of order $p_i^{e_i}$ and $p_1, \cdots, p_r$ distinct primes. Then there is an exact sequence of groups*:

$$0 \to \mathrm{Gal}(R, G) \xrightarrow{\alpha} \mathrm{Im}(\pi) \xrightarrow{\beta} \mathrm{Cont}(\mathrm{Spec}(R), (Z/2Z)^r).$$

The image of β is the set of functions f satisfying $f(p)_i = 0$ whenever $p_i \in p$ or ϕ is degenerate on G_i.

The rest of this section is devoted to proving the result just stated. The preliminary results will, however, not use the fact that G is cyclic; rather, the commutativity of $\pi(A)$ will be a sufficient assumption. We shall focus our attention on two types of gradings for commutative Galois extensions in $\mathrm{Im}(\pi)$. One is the trivial grading. The other is called the *identity grading* and is described as follows:

(4.2) Let R have no idempotents but 0 and 1. Assume that G is a finite abelian group, that its order is a unit in R, and that ϕ is symmetric and nondegenerate. Let S be a commutative Galois extension of R with group G. The nondegeneracy of ϕ implies that R contains a primitive mth root of unity, where m is the exponent of G. Since the order of G is also a unit in R, S decomposes as $S = \bigoplus_{\sigma\in G} S_\sigma$, with

$$S_\sigma = \{s \in S \mid \tau(s) = \phi(\sigma, \tau)s \text{ for } \tau \text{ in } G\},$$

a rank one projective R-module; this is proved as Theorem 1 of [5], using the orthogonality relations (3.15). The S_σ's satisfy $S_\sigma S_\tau = S_{\sigma\tau}$, and the nondegeneracy of ϕ then implies that S is a fully graded Azumaya R-algebra (separability is a consequence of S being a Galois extension). Moreover, $\pi(S) = S$ since S is commutative, and the action of G on $\pi(S)$ (defined in (3.2)) agrees with the original action of G on S.

A commutative graded Galois extension S will be said to be identity graded if $S = \bigoplus_{\sigma\in G} S_\sigma$, with S_σ homogeneous of degree σ and $\sigma(s) = \phi(\sigma, \tau)s$ for s in S_τ and all σ, τ in G.

If A is an Azumaya R-algebra, it is R-separable, and therefore separable over its ungraded center Z [1, Theorem 2.3]. Thus A is an Azumaya Z-algebra and Z is therefore a Z-direct summand of A [1, Proposition 1.2]. Moreover, $Z = \bigoplus_{\sigma\in G} Z_\sigma$, with $Z_\sigma = A_\sigma \cap Z$. Since A is a projective R-module, so is each Z_σ.

We shall assume throughout the next proof that G is a finite abelian group. However, the full force of this assumption is not needed for each of the desired conclusions. This is mentioned in the comments following (4.3).

(4.3) **Theorem.** *Let R have no idempotents but 0 and 1. Let A be an* **Azumaya** *R-algebra with ungraded center Z. Let $H = \{\sigma \in G \mid Z_\sigma \neq 0\}$; $H^\perp = \{\sigma \in G \mid \phi(\sigma, \tau) = 1$ for all τ in $H\}$.*

(a) *H is a subgroup of G and the order of H is a unit in R. If R is local then Z is a crossed product RH_f, for $f: H \times H \to U(R)$ a suitable cocycle,*

(b) *Suppose A is fully graded and $\pi(A)$ is commutative. Let $Z(A_1)$ denote the center of A_1. The G-action on $\pi(A)$ induces a H-action (resp. $H^\perp$-action) making Z (resp. $Z(A_1)$) an identity graded (resp. trivially graded) Galois extension of R. ϕ is nondegenerate on H, $G = H \times H^\perp$, and the multiplication map $Z \otimes_R Z(A_1) \to \pi(A)$ is an isomorphism of graded R-algebras and Galois extensions.*

(c) *Under the assumptions in* (b), ${}^{Z}A$, *the graded commutator of* Z *in* A, *is an Azumaya* R*-algebra, graded by* $H^{\perp}$. *The multiplication map yields an isomorphism of graded* R*-algebras*, $\pi(A) \cong Z \otimes_R {}^{Z}A$. *If* ϕ *is symmetric*, ${}^{Z}A$ *is an* $H^{\perp}$*-graded* Azumaya R*-algebra.*

Proof. Since R has no nontrivial idempotents, each Z_σ is a faithful projective R-module of finite type. Thus, for p a maximal ideal of R, $pZ_\sigma \neq Z_\sigma$, and the condition $Z_\sigma \neq 0$ is equivalent to $(Z/pZ)_\sigma \neq 0$. Thus, it suffices to prove (a) under the assumption that R is a field, since if $\mathrm{card}(H)$ is a unit in R/p for each p, it is a unit in R; that A/pA is an Azumaya R-algebra with ungraded center Z/pZ is a consequence of (2.2) and of [1, Proposition 1.4].

Let R be a field. By (2.5), A is an Azumaya R-algebra if and only if it has center R, and no homogeneous two-sided ideals. Following Knus [14, p. 126], we have that if x, y are nonzero elements of hZ and $xy = 0$, the homogeneous ideal generated by $\{a \in hA \mid ay = 0\}$ is a nonzero, proper, homogeneous two-sided ideal. Thus H is closed under multiplication.

Continue to assume that R is a field. Note that $Z_1 = R$. Choose a nonzero element x_σ in Z_σ and assume $\sigma^m = 1$; then x_σ^m is in R, hence $Z_\sigma = Z_\sigma x_\sigma^m \subseteq Z_1 x_\sigma \subseteq Z_\sigma$. This implies each Z_σ is of rank 1 over R. Hence, for σ, τ in H, $x_\sigma x_\tau = f(\sigma, \tau) x_{\sigma\tau}$ for a suitable cocycle $f\colon H \times H \longrightarrow U(R)$. This shows that $Z \cong RH_f$, a crossed product; this argument is taken from [14, Theorem 3.1]. Now Z is R-separable by the tower lemma [1, Theorem 2.3]; but a crossed product RH_f is R-separable if and only if the order of H is a unit in R [9, Lemma 4].

Let R now be any commutative ring without nontrivial idempotents. We claim that $Z_\sigma Z_\tau = Z_{\sigma\tau}$ for σ, τ in H. Each Z_σ is a projective R-module having a well-defined rank. This rank is in fact one for σ in H, since for each maximal ideal p of R, Z_σ/pZ_σ is a one-dimensional R/p-space, as shown above. The multiplication map $m\colon Z_\sigma \otimes Z_\tau \longrightarrow Z_{\sigma\tau}$ induces an *onto* map $(Z_\sigma/pZ_\sigma) \otimes (Z_\tau/pZ_\tau) \longrightarrow Z_{\sigma\tau}/pZ_{\sigma\tau}$ for each maximal ideal p of R; this implies that $p(Z_{\sigma\tau}/\mathrm{Im}(m)) = Z_{\sigma\tau}/\mathrm{Im}(m)$ and that, for some r_p in $R - p$, $r_p(Z_{\sigma\tau}/\mathrm{Im}(m)) = 0$ [2, Lemma 5.8, p. 69]. The annihilator of $Z_{\sigma\tau}/\mathrm{Im}(m)$ is therefore not contained in any maximal ideal, and m is thus onto.

If R is local each Z_σ is a free R-module with generator x_σ, σ in H. Since $Z_\sigma Z_\tau = Z_{\sigma\tau}$, it follows that $x_\sigma x_\tau = f(\sigma, \tau) x_{\sigma\tau}$ with $f(\sigma, \tau)$ a unit in R, and f a cocycle. This proves (a).

Let $B = \bigoplus_{\sigma \in H} A_\sigma$. Since $Z_\sigma Z_\tau = Z_{\sigma\tau}$, it is easily verified that the multiplication map $Z_\sigma \otimes_R A_1 \longrightarrow A_\sigma$ is onto for σ in H. This map is in fact an isomorphism: to check this, it suffices to assume R is a field. But then $Z_\sigma = Rx_\sigma$, $A_\sigma = A_1 x_\sigma$ and no element of A_1 annihilates x_σ, since A is graded simple. It follows that A_1 and A_σ have the same dimension, and $Z_\sigma \otimes_R A_1 \longrightarrow A_\sigma$ is an isomorphism. Hence we may identify $B = ZA_1 = Z \otimes_R A_1$.

Now assume A is fully graded. Then A_1 is R-separable by (2.11), hence B is R-separable. Since $B = ZA_1$, and since $\pi(A)$ is the ungraded commutator of A_1 in A, $\pi(A)$ is the ungraded commutator of B in A. By [12, Theorem 2], B is the ungraded commutator of $\pi(A)$ in A. If $\pi(A)$ is commutative, $\pi(A)$ must be contained in B. It is easily seen that in fact $\pi(A)$ is the center of B. Since $Z \otimes_R A_1 \to B$ is an isomorphism, it follows from [1, Proposition 1.4] that the multiplication map determines an isomorphism $Z \otimes_R Z(A_1) \to \pi(A)$.

We can now show that ϕ is nondegenerate on H, i.e. $H \cap H^\perp = 1$. The G-action on $\pi(A)$, defined in (3.2), is such that H acts trivially on $Z(A_1)$, since $Z_{\sigma^{-1}}Z_\sigma = Z_1$. By definition of $H^\perp$, this group acts trivially on Z. Thus, an element of $H \cap H^\perp$ acts trivially on $\pi(A)$, and must be 1. It follows that the homomorphism $\phi'\colon G \to H^*$, defined by $\phi'(\sigma)(\tau) = \phi(\sigma, \tau)$ is onto, and an isomorphism onto H^* when restricted to H. Since the kernel of ϕ is H, we may conclude that $G = H \times H^\perp$. The discussion above, and that of (3.14), may be applied to yield that Z is a Galois extension of R with group H. Moreover, $H^\perp$ acts as a group of R-algebra automorphisms of $Z(A_1)$, and the multiplication map preserves the action of $G = H \times H^\perp$. To prove that $Z(A_1)$ is a Galois extension of R with group $H^\perp$, it will suffice to prove that $Z(A_1) = \pi(A)^H$ [4, Theorem 2.2].

That $Z(A_1)$ is included in $\pi(A)^H$ follows from the definition of the action (in (3.2)) and from the fact that $Z_{\sigma^{-1}}Z_\sigma = R$. Suppose that x is a homogeneous element in $\pi(A)^H$. Since the multiplication map $Z \otimes_R Z(A_1) \to \pi(A)$ is an isomorphism, x is in $\pi(A)_\tau$ for some τ in H. The equations $\sigma x = \phi(\sigma, \tau)x = x$ may be summed over σ in H, yielding $(\Sigma_\sigma \phi(\sigma, \tau))x = nx$, where n is the order of H. The above comments concerning H and H^*, and the orthogonality relations (3.15), allow us to conclude that $nx = 0$ unless $\tau = 1$. Since n is a unit in R, τ must be 1. This concludes the proof of (b).

Let $C = \bigoplus_{\sigma \in H^\perp} A_\sigma$. Then $C \subseteq {}^ZA$. The multiplication map $Z \otimes_R C \to A$ is onto, because $G = H \times H^\perp$ and $Z_\sigma Z_\tau = Z_{\sigma\tau}$; in fact this map is an isomorphism. To prove this, it suffices to show that the multiplication map $Z \otimes_R {}^ZA \to A$ is an isomorphism. This is true modulo each maximal ideal of R by Knus's argument in [14, Theorem 3.1]; we have shown in (b) that the hypothesis needed in Knus's proof, viz. that ϕ is nondegenerate on H, holds. It follows that $Z \otimes_R {}^ZA \to A$ is an isomorphism, and that ${}^ZA = \bigoplus_{\sigma \in H^\perp} A_\sigma$.

Since $Z \otimes_R {}^ZA$ is an Azumaya Z-algebra, ${}^ZA$ is an Azumaya R-algebra [2, Corollary 2.9 (b), p. 93]. It is easily computed that $C^C = R$. If ϕ is symmetric, C is a central R-algebra.

Remarks. (a) Some of the assumptions used above can be weakened. By a localization argument, it can be shown that connectedness of R is not necessary in order for the multiplication maps $Z \otimes_R Z(A_1) \to \pi(A)$ and $Z \otimes_R {}^Z A \to A$ to be isomorphisms. If one wishes to treat Azumaya algebras for which the grading group G is not necessarily finite, or abelian (as Knus does in [14]), one can still obtain some of the results of (4.3). For example, the condition $Z_\sigma Z_\tau = Z_{\sigma\tau}$ implies that H is a central subgroup of G, and since H is finite because Z is an R-module of finite type the arguments used before apply more generally.

(b) DeMeyer has shown that if G is a finite cyclic group, any Galois extension of R with group G is commutative [7, Theorem 11]. We shall use this result freely below.

Proof of (4.1). First define α: $\mathrm{Gal}(R, G) \to \mathrm{Im}(\pi)$: Let S be a Galois extension of R with group G; S is commutative since G is cyclic. Let $D = D(S, G)$ be the trivial crossed product of S with G; i.e. $D = \bigoplus_{\sigma\in G} Su_\sigma$, with multiplication given by $(su_\sigma)(tu_\tau) = s\sigma(t)u_{\sigma\tau}$. Theorem 1.3(c) of [4] implies that S is a projective R-module of finite type, and that there is an R-algebra isomorphism $D \cong \mathrm{End}_R(S)$. Consequently D is an Azumaya R-algebra [1, Proposition 5.1], and in particular separable. We may grade D by $D_\sigma = Su_\sigma$. It is easily seen that su_σ is in D^D if and only if $s(t - \sigma(t)) = 0$ for all t in S. It follows, because S is a Galois extension [4, Theorem 1.3(f)], that $s = 0$ or $\sigma = 1$. Hence $\tau(s) = s$ for all τ in G, so that $D^D = R$. Then D is an Azumaya R-algebra. It is easily computed that $\pi(D) = Su_1 = S$. Thus $\alpha(S) = \pi(D) = S$ gives a well-defined map which is clearly one-one.

Let σ_i be a generator of G_i. We now define β. Let A be a fully graded Azumaya R-algebra, with ungraded center Z. For $\mathfrak{p}$ in $\mathrm{Spec}(R)$, $A_\mathfrak{p}$ is a fully graded Azumaya $R_\mathfrak{p}$-algebra with ungraded center $Z_\mathfrak{p}$ ((2.2); [2, Corollary 2.9, p. 93]). Let $H_\mathfrak{p}$ be the subgroup of G consisting of those σ for which $(Z_\mathfrak{p})_\sigma \neq 0$. Define $\beta(\pi A)(\mathfrak{p})_i$ to be 1 or 0 depending upon whether σ_i is in $H_\mathfrak{p}$ or not; equivalently, $\beta(\pi A)(\mathfrak{p})_i = \mathrm{rank}_{R_\mathfrak{p}} [(Z_{\sigma_i})_\mathfrak{p}]$, and $\beta(A)$ is therefore a continuous map from $\mathrm{Spec}(R)$ to $\Pi Z/2Z$. That β is well defined follows from the observation that two equivalent algebras have the same ungraded center ((1.5), [2, Corollary 2.6, p. 91], [1, Proposition 5.1]). Suppose $\beta(\pi A)(\mathfrak{p})_i = 1$; then G_i is included in $H_\mathfrak{p}$. Since by (4.2) the order of $H_\mathfrak{p}$ is a unit in $R_\mathfrak{p}$, so is the order of G_i, and consequently p_i is not in $\mathfrak{p}$. Again by (4.2), ϕ is nondegenerate on $H_\mathfrak{p}$; since $H_\mathfrak{p} = G_i \times L_i$, with the orders of G_i and L_i relatively prime, ϕ is nondegenerate on G_i. Thus, the image of β is contained in the indicated set.

It is easily seen that $\beta\alpha = 0$, since for S a commutative Galois extension of R with group G, $D(S, G)$ has ungraded center R. Conversely, if $\beta(\pi A) = 0$, then the ungraded center Z of A is R, since $(Z_\sigma)_\mathfrak{p} = 0$ for $\mathfrak{p}$ in $\mathrm{Spec}(R)$ and $\sigma \neq 1$ in G. By (4.2), $\pi(A) = Z(A_1)$, a trivially graded Galois extension. Consequently, $\pi(A) = \alpha(\pi A)$. The sequence is thus exact.

We now show that the image of β is as claimed. Fix an index i and suppose ϕ is nondegenerate on G_i. Let f be a continuous map from $\operatorname{Spec}(R)$ to $\Pi_{i=1}^r Z/2Z$ with $f(\mathfrak{p}) = 0$ for $j \neq i$ and $f(\mathfrak{p})_i = 0$ or 1 depending on whether p_i is in $\mathfrak{p}$ or not. Then ([3, p. 130])

$$\operatorname{Spec}(R) = \{\mathfrak{p} \mid f(\mathfrak{p})_i = 1\} \cup \{\mathfrak{p} \mid f(\mathfrak{p})_i = 0\},$$

a disjoint union of open and closed sets. This leads to a decomposition $R = S \times T$, with $\operatorname{Spec}(S) = \{\mathfrak{p} \mid f(\mathfrak{p})_i = 1\}$. Consider the group ring SG_i; first of all p_i is not in any element of $\operatorname{Spec}(S)$ and is therefore a unit in S. Coupled with the fact that ϕ is nondegenerate on G_i, this implies that the orthogonality relations hold between G_i and G_i^* (cf. (3.15)) and that the center of SG_i is S (the grading is by G_i). Because the order of G_i is a unit in S, SG_i is S-separable [21, Theorem 1.1], and is thus an Azumaya S-algebra.

Let $H_i = \Pi_{j \neq i} G_j$, so that $G = G_i \times H_i$. Let T_i be any commutative Galois extension of T with group H_i; e.g. $T_i = H_i T$, the algebra of set functions from H_i to T (cf. [10, §1, p. 68]). By the argument used previously in this proof, $D_i = D(T_i, H_i)$ is a H_i-graded Azumaya T-algebra, with ungraded center T.

Let $C = SG_i \times D_i$, an R-algebra since $R = S \times T$. The formula $C_{\sigma,\tau} = (SG_i)_\sigma \times (D_i)_\tau$ defines a $G_i \times H_i$-grading, i.e. a G-grading on C. That C is R-separable is a result about ungraded algebras [2, Proposition 2.20, p. 99]. A computation shows that C is R-central, and thus an Azumaya R-algebra. The ungraded center of C is $SG_i \times T$. Now, C is equivalent to a fully graded Azumaya R-algebra E whose ungraded center is that of C (this was shown previously in this proof). Then $\beta(\pi E) = f$, and the image of β is as claimed in the statement of the theorem.

To complete the proof it suffices to prove that α and β are homomorphisms. Let S and T be Galois extensions of R with group G. The product of S and T in $\operatorname{Gal}(R, G)$ is $S \cdot T = (S \otimes_R T)^{\Delta G}$, the set of elements of $S \otimes_R T$ fixed by $\{(\sigma, \sigma^{-1}) \mid \sigma \text{ in } G\}$. Write A for $D(S, G)$, B for $D(T, G)$. The multiplication in $\operatorname{Im}(\pi)$ is such that $\pi(A) \cdot \pi(B) = \pi(A \otimes B)$ (cf. (3.13)). But $\pi(A)$ and $\pi(B)$, i.e. S and T, are trivially graded. By (3.6), we conclude that α is a homomorphism.

Since β being a homomorphism depends on the behavior of our algebras at localizations, we shall assume that R is local. Let E be a fully graded Azumaya R-algebra with ungraded center $Z(E)$, H the set of σ in G with $Z(E)_\sigma \neq 0$. We obtain from (4.3) that $Z(E) = \pi(E)^H$ and $Z(E_1) = \pi(E)^{H^\perp}$, with $G = H \times H^\perp$. Fix an index i and let $G_i = \langle\sigma\rangle$. Now $Z(E)_\sigma = 0$ is equivalent to $\langle\sigma\rangle$ not being a subgroup of H, which is in turn equivalent to $\langle\sigma\rangle$ being a subgroup of $H^\perp$ since G is cyclic. We therefore conclude:

(A) $Z(E)_\sigma = 0$ (resp. $\neq 0$) if and only if $\pi(E)^\sigma$ is included in $Z(E_1)$ (resp. $Z(E)$).

Let A and B be fully graded Azumaya R-algebras (with R still local and $G_i = \langle\sigma\rangle$), and let $C = A \otimes B$. We must relate $Z(A)_\sigma$ and $Z(B)_\sigma$ being nonzero to $Z(C)_\sigma$ being nonzero, and to this end we shall examine $\pi(C)$ more closely. Formula (*) of (3.6) allows us to identify $\pi(C)$ as a subset of $\pi(A) \otimes \pi(B)$. We may rephrase this formula by saying that $\pi(C) = [\pi(A) \otimes \pi(B)]^{\Delta G}$, where $G \times G$ acts on $\pi(A) \otimes \pi(B)$ via $(\theta, \tau)(a \otimes b) = \phi(\tau^{-2}, b)\theta(a) \otimes \tau(b)$, and ΔG is the kernel of the multiplication map from $G \times G$ to G. The G-action on $[\pi(A) \otimes \pi(B)]^{\Delta G}$ compatible with that on $\pi(C)$ is $\tau(a \otimes b) = a \otimes \tau(b) = \phi(\tau^2, b)\tau(a) \otimes b$. We therefore obtain that $\pi(C)^\sigma = [\pi(A) \otimes \pi(B)]^{(\Delta G)\cdot\sigma}$. Now $\pi(A)^\sigma \otimes \pi(B)^\sigma$ is included in $[\pi(A) \otimes \pi(B)]^\sigma$. Let $K = \Pi_{j \neq i} G_j$, so that $\langle\sigma\rangle \times K = G$. A computation shows that $[\pi(A)^\sigma \otimes \pi(B)^\sigma]^{\Delta K}$ is included in $[\pi(A) \otimes \pi(B)]^{(\Delta G)\cdot\sigma}$, and since both algebras are Galois extensions of R with group K, this inclusion is an equality [4, Theorem 3.4]. We therefore have

(B) $$\pi(C) = [\pi(A)^\sigma \otimes \pi(B)^\sigma]^{\Delta K}$$

To complete the proof that β is a homomorphism, i.e. that $\beta(\pi(A)\cdot\pi(B))(\mathfrak{p})_i = \beta(\pi A)(\mathfrak{p})_i + \beta(\pi B)(\mathfrak{p})_i$, we must examine four cases.

Case (0-0). If $Z(A)_\sigma = 0 = Z(B)_\sigma$ then $\pi(A)^\sigma \subset Z(A_1)$ and $\pi(B)^\sigma \subset Z(B_1)$ by (A), whence $Z(C)^\sigma \subset Z(C_1)$ by (B).

Case (0-1). If $Z(A)_\sigma = 0$ and $Z(B)_\sigma \neq 0$ then the grading on $\pi(C)^\sigma$ comes from that on $\pi(B)^\sigma$. For τ in G_i, a in $\pi(A)^\sigma$ and b in $\pi(B)^\sigma$ we have $\tau(a \otimes b) = a \otimes \tau(b) = \phi(\tau, b)a \otimes b$ (since $Z(B)$ is identity graded). Hence $\pi(C)^\sigma$ is identity graded and contained in $Z(C)$.

Case (1-0). If $Z(A)_\sigma \neq 0$ and $Z(B)_\sigma = 0$, a computation similar to the above shows that $Z(C)_\sigma \neq 0$.

Case (1-1). If $Z(A)_\sigma \neq 0 \neq Z(B)_\sigma$ then an element $a_\theta \otimes b_\tau$ of $\pi(C)^\sigma$ satisfies

$$\begin{aligned} a_\theta \otimes b_\tau &= \phi(\sigma^2, \tau)\sigma(a_\theta) \otimes \sigma^{-1}(b_\tau) \\ &= \phi(\sigma^2, \tau)\phi(\sigma, \theta)\phi(\sigma^{-1}, \tau)(a_\theta \otimes b_\tau) \\ &= \phi(\sigma, \theta)\phi(\sigma, \tau)(a_\theta \otimes b_\tau). \end{aligned}$$

Since ϕ is nondegenerate on G_i, $\theta = \tau^{-1}$ and $\pi(C)^\sigma$ is trivially graded. This completes the proof of (4.1).

(4.5) **Example.** The group $B(R, G)$ need not be abelian. We shall in fact see that even $\operatorname{Im}(\pi)$ need not be abelian.

Let $G = Z/3Z$ and let σ be a generator of G. Let R contain $1/3$, a primitive cube root of unity ξ and a unit b which is not a cube in R. Let a be a unit in R. Choose a bilinear map ϕ satisfying $\phi(\sigma, \sigma) = \xi$. Let $S = R \oplus Ru \oplus Ru^2$ with $u^3 = a$, $T = R \oplus Rv \oplus Rv^2$ with $v^3 = b$. View S as fully graded by G, T as trivially graded. Each of S and T is in $\mathrm{Im}(\pi)$, since $S = \pi(S)$ and $T = \pi(D(T, G))$.

We have that $S \cdot T = (S \otimes T)^{\Delta G}$ in $\mathrm{Im}(\pi)$, where $G \times G$ acts on $S \otimes T$ by $(\sigma, \tau)(a \otimes b) = \phi(\tau, b)\sigma(a) \otimes \tau(b)$. One verifies that

$$S \cdot T = R \oplus R(u \otimes v) \oplus R(u \otimes v)^2,$$

with $\sigma(u \otimes v) = \xi(u \otimes v)$ and $(u \otimes v)^3 = ab$, whereas

$$T \cdot S = R \oplus R(v^2 \otimes u) \oplus R(v^2 \otimes u)^2,$$

with $\sigma(v^2 \otimes u) = \xi(v^2 \otimes u)$ and $(v^2 \otimes u)^3 = ab^2$. Since $S \cdot T$ and $T \cdot S$ are both Galois extensions with normal bases, and R possesses a cube root of unity and $1/3$, $S \cdot T$ and $T \cdot S$ are isomorphic if and only if ab and ab^2 represent the same element in $H^2(G, U(R)) \cong U(R)/U(R)^3$ [5, Remark 3]. Since b is not a cube in R, we have that $S \cdot T \neq T \cdot S$ in $\mathrm{Im}(\pi)$.

We are able to describe $\mathrm{Im}(\pi)$ intrinsically for G any finite abelian group and ϕ trivial. In this case $\mathrm{Im}(\pi)$ is $\mathrm{Gal}(R, G)$, the group of ungraded Galois extensions of R with group G. As the next example shows, however, we cannot expect to describe $\mathrm{Im}(\pi)$ as easily for general G as for finite cyclic G.

(4.6) **Example.** We shall show that the technique of splitting off the ungraded center (cf. (4.3)) does not necessarily work if $\pi(A)$ is not commutative. We shall also show that the conclusion of [4, Theorem 3.1], viz. that an Azumaya R-algebra decomposes as $RH_f \otimes_R M$, with M a matrix ring, may hold even if the hypotheses of that theorem are not satisfied.

Let Q be the field of rationals, $R = Q(i)$, c a nonsquare in R and $Z = R(\alpha)$, with $\alpha^2 = c$. Let A (resp. B) be the generalized quaternion algebra over Z generated by u and v, with $u^2 = \alpha$, $uv = -vu$ and $v^2 = \alpha$ (resp. $v^2 = c\alpha$). Let $G = Z/4Z \times Z/2Z$ with σ and τ generators of $Z/4Z$ and $Z/2Z$ respectively. Let C stand for either A or B; grade C by setting $C_\sigma = Ru$, $C_{\sigma\tau} = Rv$. Define ϕ by $\phi(\sigma, \sigma) = i$, $\phi(\sigma, \tau) = \phi(\tau, \sigma) = \phi(\tau, \tau) = -1$. A computation shows that C is central over R. C is a central simple Z-algebra (since it is a generalized quaternion algebra), and is therefore separable over Z. In turn Z is separable over R [11, Corollary 2.4] and, by the tower lemma, C is separable over R. Hence, C is an Azumaya R-algebra. Now $\pi(C) = C$ since $C_1 = R$; thus C is a graded noncommutative Galois extension of R with group G. The action of G is determined by the formulas $\sigma(u) = iu$, $\sigma(v) = iv$, $\tau(u) = u$, $\tau(v) = -v$. Now Z is graded by $Z = C_1 \oplus C_{\sigma^2}$, hence $H = \{1, \sigma^2\}$ (cf. (4.3)). It is clear that H is contained in $H^\perp$, so that the hypothesis of Knus's theorem, viz. that ϕ is nondegenerate on H, does not hold. The multiplication map $Z \otimes_R {}^ZC \to C$ is neither one-one nor onto (a computation shows that $Z_C = C_1 \oplus C_{\sigma^2\tau} \oplus C_{\sigma^2} \oplus C_\tau$).

However, A is isomorphic to the algebra of 2×2 matrices over Z, with u

and v corresponding to the respective matrices $\begin{pmatrix} 0 & ia \\ -i & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}$. It is not difficult to show that B is *not* isomorphic to this matrix algebra over Z; one uses the fact that a 2×2 matrix is similar to one with a zero diagonal entry and pursues a straightforward computation based on the equations $u^2 = \alpha$, $v^2 = c\alpha$, $uv = -vu$.

For R a field let $\mathcal{G}_R$ be the abelianized Galois group of R. For G cyclic, $\mathrm{Gal}(R, G) \cong \mathrm{Hom}_c(\mathcal{G}_R, G)$ [10, Theorem 8]. Thus, our exact sequences provide a connection between the Brauer group of R and the fundamental group of R.

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DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK AT ALBANY, ALBANY, NEW YORK 12203 (Current address of L. N. Childs)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS 61801

DEPARTMENT OF MATHEMATICS, QUEEN'S UNIVERSITY, KINGSTON, ONTARIO, CANADA (Current address of M. Orzech)

Current address (G. Garfinkel): Department of Mathematics, Oakland University, Rochester, Michigan 48063